Hint for HW 2 Q 1

$$s = A f B$$

$$s(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} A(\alpha,x)\, f(x,y)\, B(y,\beta)$$

$$= \sum_{y=1}^{N}\left(\sum_{x=1}^{N} A(\alpha,x)\, B(y,\beta)\, f(x,y)\right)$$

$$= A(\alpha,1)B(1,\beta)f(1,1) + \cdots + A(\alpha,N)B(1,\beta)f(N,1)$$
$$+ A(\alpha,1)B(2,\beta)f(1,2) + \cdots + A(\alpha,N)B(2,\beta)f(N,2)$$
$$+ \cdots$$
$$+ A(\alpha,1)B(N,\beta)f(1,N) + \cdots + A(\alpha,N)B(N,\beta)f(N,N)$$

$$= \underbrace{\left[\, \vec{a}_\alpha B(1,\beta) \;\middle|\; \vec{a}_\alpha B(2,\beta) \;\middle|\; \cdots \;\middle|\; \vec{a}_\alpha B(N,\beta) \,\right]}_{\text{a row with length } N^2} \begin{bmatrix} \vec{f}_1 \\ \hline \vdots \\ \hline \vec{f}_N \end{bmatrix}$$

$\vec{a}_\alpha$: α-th row of A

$\vec{f}_1$: 1^{st} col of f

a row with length N^2

Block Matrices

$$\begin{bmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{N1} \\ \vdots \\ s_{1N} \\ \vdots \\ s_{NN} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} \vec{a}_1 B(1,1) \\ \vec{a}_2 B(1,1) \\ \vdots \\ \vec{a}_N B(1,1) \end{matrix}} & \boxed{\begin{matrix} \vec{a}_1 B(2,1) \\ \vec{a}_2 B(2,1) \\ \vdots \\ \vec{a}_N B(2,1) \end{matrix}} & \cdots & \boxed{\begin{matrix} \vec{a}_1 B(N,1) \\ \vec{a}_2 B(N,1) \\ \vdots \\ \vec{a}_N B(N,1) \end{matrix}} \\ \vdots & & & \vdots \\ \boxed{\begin{matrix} \vec{a}_1 B(1,N) \\ \vdots \\ \vec{a}_N B(1,N) \end{matrix}} & \boxed{\begin{matrix} \vec{a}_1 B(2,N) \\ \vdots \\ \vec{a}_N B(2,N) \end{matrix}} & \cdots & \boxed{\begin{matrix} \vec{a}_1 B(N,N) \\ \vdots \\ \vec{a}_N B(N,N) \end{matrix}} \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{N1} \\ \vdots \\ f_{1N} \\ \vdots \\ f_{NN} \end{bmatrix}$$

DFT

$g \in \mathbb{C}^{M \times N}$, using zero-indexing: $\begin{bmatrix} g(0,0) & \cdots & g(0,N-1) \\ \vdots & & \vdots \\ g(M-1,0) & \cdots & g(M-1,N-1) \end{bmatrix}$

The 2D DFT is defined as:

$$0 \le m \le M-1, \quad 0 \le n \le N-1$$

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

$$= \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \underbrace{\left(\frac{1}{M} e^{-j2\pi\frac{km}{M}}\right)}_{\text{all about row}} g(k,l) \underbrace{\left(\frac{1}{N} e^{-j2\pi\frac{ln}{N}}\right)}_{\text{all about column}}$$

write: $U_M(m,k)$ $\quad U_N(l,n)$

$\therefore$ Separable, $\hat{g} = U_M\, g\, U_N$

write $w_M = e^{-j2\pi\frac{1}{M}}$, $w_N = e^{-j2\pi\frac{1}{N}}$

$$U_M = \frac{1}{M}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_M^1 & w_M^2 & \cdots & w_M^{M-1} \\ \vdots & w_M^2 & w_M^4 & \cdots & w_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_M^{M-1} & w_M^{2(M-1)} & \cdots & w_M^{(M-1)^2} \end{bmatrix}$$

and

$$U_N = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_N^1 & w_N^2 & \cdots & w_N^{N-1} \\ \vdots & w_N^2 & w_N^4 & \cdots & w_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_N^{N-1} & w_N^{2(N-1)} & \cdots & w_N^{(N-1)^2} \end{bmatrix}$$

$$\tilde{U}_M := \sqrt{M}\, U_M$$

$$= \frac{1}{\sqrt{M}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_M^1 & w_M^2 & \cdots & w_M^{M-1} \\ \vdots & w_M^2 & w_M^4 & \cdots & w_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_M^{M-1} & w_M^{2(M-1)} & \cdots & w_M^{(M-1)^2} \end{bmatrix}$$

Then $\tilde{U}_M$ is unitary ($\tilde{U}_M \cdot \tilde{U}_M^* = I$)

$$\left[\tilde{U}_M \cdot \tilde{U}_M^*\right](\alpha, \beta)$$

$$= \sum_{\gamma=0}^{M-1} \tilde{U}_M(\alpha,\gamma)\left[\tilde{U}_M^*\right](\gamma,\beta)$$

$$= \sum_{\gamma=0}^{M-1} \tilde{U}_M(\alpha,\gamma)\,\overline{\tilde{U}_M(\beta,\gamma)}$$

$$= \frac{1}{M}\sum_{\gamma=0}^{M-1} e^{-j2\pi\left(\frac{\alpha\gamma}{M}\right)} \cdot e^{j2\pi\left(\frac{\beta\gamma}{M}\right)}$$

$$= \frac{1}{M} \sum_{r=0}^{M-1} e^{j2\pi \frac{(\alpha-\beta)}{M} r}$$

$$= \begin{cases} \frac{1}{M} \sum_{r=0}^{M-1} 1 & , \text{ if } \alpha = \beta \\ \frac{1}{M} \frac{e^{j2\pi \frac{(\alpha-\beta)}{M} M} - 1}{e^{j2\pi \frac{(\alpha-\beta)}{M}} - 1} & , \text{ if } \alpha \neq \beta \end{cases}$$

$$= \begin{cases} 1 & \text{ if } \alpha = \beta \\ \frac{1}{M} \frac{e^{j2\pi (\alpha-\beta)} - 1}{e^{j2\pi (\frac{\alpha-\beta}{M})} - 1} & , \text{ if } \alpha \neq \beta \end{cases}$$

non-zero integer

$$= \begin{cases} 1 & , \text{ if } \alpha = \beta \\ 0 & , \text{ if } \alpha \neq \beta \end{cases}$$

$\therefore$ $\tilde{U}_M \cdot \tilde{U}_M^*$ is Identity.

Similarly, $\tilde{U}_N$ is also unitary.

Assuming periodicity,

if we shift an image,

$$\tilde{g}(k,l) = g(\underbrace{k-k_0}_{\text{shift down by } k_0 \text{ units}}, \underbrace{l-l_0}_{\text{shift right by } l_0 \text{ units.}}) \quad , g \in \mathbb{C}^{N \times N}$$

Then the DFT of $\tilde{g}$ is given by:

$$\hat{\tilde{g}}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \tilde{g}(k,l)\, e^{-j\frac{2\pi}{N}(km+ln)}$$

$$= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k-k_0, l-l_0)\, e^{-j\frac{2\pi}{N}(km+ln)}$$

$$= \frac{1}{N^2} \sum_{k=-k_0}^{N-1-k_0} \sum_{l=-l_0}^{N-1-l_0} g(k,l)\, e^{-j\frac{2\pi}{N}((k+k_0)m+(l+l_0)n)}$$

if $l_0 > 0$,

$$\frac{e^{-j\frac{2\pi}{N}(k_0m+l_0n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \left(\sum_{l=0}^{N-1-l_0} g(k,l)\exp(\cdots) + \sum_{l=-l_0}^{-1} g(k,l)\exp(\cdots) \right)$$

$$= \frac{e^{-j\frac{2\pi}{N}(k_0m+l_0n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \left(\sum_{l=0}^{N-1-l_0} g(k,l)\exp(\cdots) - \sum_{l=N-l_0}^{N-1} g(k,l-N)\exp(\cdots) \exp\left(-j\frac{2\pi}{N}ln + j\frac{2\pi}{N}Nn\right) \right)$$

$$= \frac{e^{-j\frac{2\pi}{N}(k_0m+l_0n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \left(\sum_{l=0}^{N-1-l_0} g(k,l)\exp(\cdots) - \sum_{l=N-l_0}^{N-1} g(k,l)\exp(\cdots) \exp\left(-j\frac{2\pi}{N}ln\right) \right)$$

$$= \frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \sum_{l=0}^{N-1} g(k,l) \exp\left(-j\frac{2\pi}{N}(km + ln)\right)$$

if $l_0 < 0$,

$$\frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \left(\sum_{l=-l_0}^{N-1} g(k,l)\exp(\cdots) + \sum_{l=N}^{N-1-l_0} g(k,l)\exp(\cdots) \right)$$

$$= \frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \left(\sum_{l=-l_0}^{N-1} g(k,l)\exp(\cdots) - \sum_{l=0}^{-l_0-1} g(k,l+N)\exp(\cdots) \exp\left(-j\frac{2\pi}{N}ln - j\frac{2\pi}{N}Nn\right) \right)$$

$$= \frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \left(\sum_{l=-l_0}^{N-1} g(k,l)\exp(\cdots) - \sum_{l=0}^{-l_0-1} g(k,l)\exp(\cdots) \exp\left(-j\frac{2\pi}{N}ln\right) \right)$$

$$= \frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \sum_{l=0}^{N-1} g(k,l) \exp\left(-j\frac{2\pi}{N}(km + ln)\right)$$

If $l_0 = 0$, nothing needs to do.

In any cases,

$$= \frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=-k_0}^{N-1-k_0} \sum_{l=0}^{N-1} g(k,l)\exp(\cdots)$$

Similarly,

$$= \frac{e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\exp(\cdots)$$

$$= \hat{g}(m,n)\, e^{-j\frac{2\pi}{N}(k_0 m + l_0 n)}$$